Tutorial 8 : Selected problems of Assignment 8

Leon Li

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Q1) (HW8, Q2) Let $A = (a_{ij})$ be a real $n \times n$ matrix,

define the 1-norm $\|A\|_1$ of A by

$\|A\|_1 := \sup\{\|Ax\|_1 : x \in \mathbb{R}^n \text{ s.t. } \|x\|_1 \leq 1\}$, where $\|x\|_1 := \sum_{k=1}^{n} |x_k|$

(a) Show that $\|A\|_1 = \inf \underbrace{\{M \in \mathbb{R} \mid \|Ax\|_1 \leqslant M\|x\|_1, \forall x \in \mathbb{R}^n\}}_{\mathcal{S}}$.

(b) Show that $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}| \ (=: \alpha)$

(c) Show that if $\|A\|_1 < 1$, then $I + A$ is invertible.

Sol: (a) $[\leqslant]$ $\forall x \in \mathbb{R}^n$ w/ $\|x\|_1 \leq 1$, $\forall M \in \mathcal{S}$, by definition

$\|Ax\|_1 \leqslant M$. $\therefore \|A\|_1 \leqslant M$. Hence $\|A\|_1 \leqslant \inf \mathcal{S}$

$[\geqslant]$ Want to show that $\forall x \in \mathbb{R}^n$, $\|Ax\|_1 \leqslant \|A\|_1 \|x\|_1$:

Case 1: $x = 0$: both sides are 0.

Case 2: $x \neq 0$: then $\left\| \frac{x}{\|x\|_1} \right\|_1 = 1$, $\therefore \left\| A\left(\frac{x}{\|x\|_1}\right) \right\|_1 \leq \|A\|_1$

$\Rightarrow \|Ax\|_1 \leqslant \|A\|_1 \|x\|_1$

$\therefore \|A\|_1 \in \mathcal{S}$, hence $\|A\|_1 \geq \inf \mathcal{S}$.

(b) [$\leq$] Want to show $\alpha \in \mathcal{S}$: $\forall x \in \mathbb{R}^n$.

$$\|Ax\|_1 = \left\| \left(\sum_{j=1}^{n} a_{1j} x_j, \dots, \sum_{j=1}^{n} a_{nj} x_j \right) \right\|_1 = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right|$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}||x_j| = \sum_j |x_j| \left(\sum_i |a_{ij}| \right) \leq \sum_k |x_k| \cdot \left(\max_j \left(\sum_i |a_{ij}| \right) \right) = \alpha \|x\|_1$$

$\therefore \alpha \in \mathcal{S}$, hence $\|A\|_1 = \inf \mathcal{S} \leq \alpha$.

[$\geq$] $\forall\, 1 \leq j \leq n$, $Ae_j = (a_{1j}, \cdots, a_{nj})$, $\therefore \|A\|_1 \geq \|Ae_j\|_1 = \sum_{i=1}^{n} |a_{ij}|$, $\therefore \|A\|_1 \geq \alpha$

(c) Applying the Perturbation of Identity: $(X, \|\cdot\|) = (\mathbb{R}^n, \|\cdot\|_1)$

let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be defined as $\Phi(x) = x + Ax$, then $\Phi(0) = 0$:

Also $\Phi = I + \Psi$, where $\Psi(x) = Ax$. Showing Ψ is a contraction:

$$\forall\, x, x' \in \mathbb{R}^n, \quad \|\Psi(x) - \Psi(x')\|_1 = \|A(x - x')\|_1 \leq \|A\|_1 \|x - x'\|_1$$

$\therefore$ Choose $\gamma = \|A\|_1 < 1$, Ψ is a contraction on $\mathbb{R}^n$.

$\therefore$ By the theorem, $\forall r > 0$, $\Phi(x) = 0$ is uniquely solvable on $\overline{B_r(0)}$

Hence Φ is uniquely solvable on $\mathbb{R}^n$.

As $\Phi(0) = 0$, $\forall x \in \mathbb{R}^n$ s.t. $\Phi(x) = 0$, $x = 0$, i.e. $\operatorname{Ker} \Phi = \{0\}$

Therefore, $I + A$ is invertible. $\square$

Q2) (HW8, Q3) (Inverse Function Theorem for $\mathbb{R}$)

Let $f : [a,b] \to \mathbb{R}$ be a C^1 function s.t. $f(a) < f(b)$. Show that f admits a global differentiable inverse $g \iff \forall x \in (a,b),\ f'(x) > 0$

Pf) [$\Rightarrow$] Assuming there exists such g, then $\forall x \in (a,b)$,

$g(f(x)) = x$; differentiating both sides with respect to x :

$g'(f(x)) \cdot f'(x) = 1$, $\therefore \forall x \in (a,b),\ f'(x) \neq 0$.

As $f' : (a,b) \to \mathbb{R}$ is continuous, either $\forall x,\ f'(x) > 0$ or $\forall x,\ f'(x) < 0$

Since $f(a) < f(b)$, by Mean Value Theorem, $\exists\, \xi \in (a,b)$ s.t.

$f'(\xi)(b-a) = f(b) - f(a) > 0$, hence $f'(\xi) > 0$.

$\therefore \forall x \in (a,b),\ f'(x) > 0$.

[$\Leftarrow$] We first show that f is strictly increasing: $\forall x, y \in [a,b]$, $x < y$

by MVT, $\exists\, \xi \in (a,b)$ s.t. $f(y) - f(x) = f'(\xi)(y-x) > 0$. $\therefore f(x) < f(y)$

$\therefore f : [a,b] \to [f(a), f(b)]$ is strictly increasing continuous function.

By Continuous Inverse Theorem ([Bartle, Thm 5.6.5]), there exists global continuous inverse $g: [f(a), f(b)] \to [a,b]$

Since f is differentiable with $f'(x) \neq 0$, $\forall x \in (a,b)$,

by [Bartle: Thm 6.1.8] (whose proof uses Carathéodory Theorem),

g is also differentiable. $\boxtimes$

Q3) (HW8, Q6) (Open mapping Theorem)

Let $F: U \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^n$ be C^1 s.t. $\forall x \in U$, $\det(DF(x)) \neq 0$

(a) Show that F is an open map: $\forall V \subseteq U$ open, $F(V) \subseteq \mathbb{R}^n$ is open.

(b) Provide an example of such F so that F has no global inverse.

Pf: (a) Given $V \subseteq U$ open, showing $F(V) \subseteq \mathbb{R}^n$ is open by definition:

$\forall y \in F(V)$, $\exists x \in V$ s.t. $y = F(x)$. Since $F|_V$ is C^1 w/ $\det(DF(x)) \neq 0$

by Inverse Function Theorem, $\exists R > 0$ s.t. $B_R(y) \subseteq F(V)$ (hence $\underset{x}{\underset{\cup}{G}} := F^{-1}(B_R(y)) \subseteq V$)

and $F: G \to B_R(y)$ is C^1 bijective w/ C^1 inverse.

Hence $y \in B_R(y) \subseteq F(V)$. Therefore, $F(V)$ is open.

(b) Define $F: (0, +\infty) \times \mathbb{R} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $F(r, \theta) = (r\cos\theta, r\sin\theta)$.

Then $\forall (r, \theta)$, $\det(DF(r, \theta)) = r \neq 0$. $\therefore$ F satisfies the assumption

However, F has no global inverse: F is not bijective as

for instance $F(1, 0) = (1, 0) = F(1, 2\pi)$.

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